Solutions to tutorial exercises for stochastic processes

T1. (a) $\Rightarrow$: Suppose $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. We can write

$$\{\tau < t\} = \bigcup_{\substack{s<t\\ s\in\mathbb{Q}}} \{\tau \leq s\} \in \mathcal{F}_t.$$

$\Leftarrow$: Suppose $\{\tau < t\} \in \mathcal{F}_t$ for all $t \geq 0$. In this case we have

$$\{\tau \leq t\} = \bigcap_{\substack{s>t\\ s\in\mathbb{Q}}} \{\tau < s\},$$

which is an element of $\mathcal{F}_s$ for all $s > t$, so that

$$\{\tau \leq t\} \in \bigcap_{\substack{s>t\\ s\in\mathbb{Q}}} \mathcal{F}_s.$$

Since the filtration is right-continuous we conclude that $\{\tau \leq t\} \in \mathcal{F}_t$.

(b) Suppose τ_n is a sequence of stopping times. We have

$$\left\{\sup_n \tau_n \leq t\right\} = \bigcap_n \{\tau_n \leq t\} \in \mathcal{F}_t,$$

and similarly

$$\left\{\inf_n \tau_n \leq t\right\} = \bigcup_n \{\tau_n \leq t\} \in \mathcal{F}_t,$$

so that $sup_n \tau_n$ and $\inf_n \tau_n$ are stopping times. For $\limsup_n \tau_n$ we can write

$$\limsup_{n\to\infty} \tau_n = \inf_{n\geq 0} \sup_{k\geq n} \tau_k =: \inf_{n\geq 0} \sigma_n.$$

By the above observations σ_n is a stopping time and thus so is $\limsup_n \tau_n$. Similarly

$$\liminf_{n\to\infty} \tau_n = \sup_{n\geq 0} \inf_{k\geq n} \tau_k,$$

is a stopping time. Finally if $\lim_{n\to\infty} \tau_n$ exists it is equal to $\liminf_{n\to\infty} \tau_n$, so that it is a stopping time as well.

T2. (a) Firstly $\Omega \in \mathcal{F}_\tau$, since τ is a stopping time. Suppose $A \in \mathcal{F}_\tau$, then $A \cap \{\tau \le t\} \in \mathcal{F}_t$ for all $t \ge 0$. Since $\mathcal{F}_t$ is a σ-algebra it holds that

$$A^c \cap \{\tau \le t\} = \big(A \cap \{\tau \le t\}\big)^c \cap \{\tau \le t\} \in \mathcal{F}_t,$$

so $A^c \in \mathcal{F}_\tau$. Lastly suppose $A_1, A_2, \dots \in \mathcal{F}_\tau$. Then since $\mathcal{F}_t$ is a σ-algebra we have

$$\left(\bigcup_{i=1}^\infty A_i\right) \cap \{\tau \le t\} = \bigcup_{i=1}^\infty \big(A_i \cap \{\tau \le t\}\big) \in \mathcal{F}_t.$$

So $\mathcal{F}_\tau$ is indeed a σ-algebra.

(b) Let $s \ge 0$. We have

$$\{\tau \le s\} \in \mathcal{F}_\tau,$$

since for any $t \ge 0$ it holds that

$$\{\tau \le s\} \cap \{\tau \le t\} = \{\tau \le s \wedge t\} \in \mathcal{F}_{s \wedge t} \subseteq \mathcal{F}_t,$$

since τ is a stopping time.

(c) Suppose $\tau_1 \le \tau_2$. Then $\{\tau_2 \le t\} \subseteq \{\tau_1 \le t\}$ for all $t \ge 0$. Suppose $A \in \mathcal{F}_{\tau_1}$ and let $t \ge 0$, then

$$A \cap \{\tau_2 \le t\} = A \cap \{\tau_1 \le t\} \cap \{\tau_2 \le t\} \in \mathcal{F}_t,$$

since $= A \cap \{\tau_1 \le t\} \in \mathcal{F}_t$ and $\{\tau_2 \le t\} \in \mathcal{F}_t$. So $\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2}$.

(d) Suppose $\tau_n \downarrow \tau$. Then by the argumentation in (b) we have $\mathcal{F}_\tau \subseteq \mathcal{F}_{\tau_n}$ for all n. So in particular $\mathcal{F}_\tau \subseteq \cap_n \mathcal{F}_{\tau_n}$. Now let $A \in \cap_n \mathcal{F}_{\tau_n}$. Then there exists some $N \in \mathbb{N}$ such that for all $n \ge N$ we have $A \cap \{\tau_n \le t\} \in \mathcal{F}_t$. Since $\mathcal{F}_t$ is a σ-algebra we find

$$\mathcal{F}_t \ni \bigcap_{n=N}^\infty A \cap \{\tau_n \le t\} = A \cap \bigcap_{n=N}^\infty \{\tau_n \le t\} = A \cap \{\tau \le t\},$$

so that $A \in \mathcal{F}_\tau$. We conclude that $\mathcal{F}_\tau = \cap_n \mathcal{F}_{\tau_n}$.

T3. (a) Let $\tau_1 = \inf\{s \ge 1 \mid B_s = 0\}$ and let $\tau = \inf\{s \ge 0 \mid B_s = 0\}$. Firstly we can write

$$\{\tau_1(\omega) \le t\} = \big\{\inf\{s \ge 1 \mid \omega(s) = 0\} \le t\big\} = \big\{\inf\{s \ge 0 \mid \omega(s+1) = 0\} + 1 \le t\big\},$$

so that

$$\mathbb{1}\{\tau_1 \le t\} = \mathbb{1}\{\tau \le t-1\} \circ \theta_1.$$

Now we can calculate the distribution of τ_1 using the Markov property:

$$\begin{aligned}\mathbb{P}^0(\tau_1 \le t) &= \mathbb{E}^0[\mathbb{1}_{\{\tau_1 \le t\}}] = \mathbb{E}^0\big[\mathbb{E}^0[\mathbb{1}_{\{\tau \le t-1\}} \circ \theta_1 | \mathcal{F}_1]\big] = \mathbb{E}^0\big[\mathbb{E}^{B_1}[\mathbb{1}_{\{\tau \le t-1\}}]\big] \\ &= \int_{-\infty}^\infty p_1(0,x)\mathbb{P}^x(\tau \le t-1)\mathrm{d}x,\end{aligned}$$

where $p_t(a,b) = (2\pi t)^{-1/2} \exp\left(\frac{-|a-b|^2}{2t}\right)$. Using the distribution of τ we get

$$\begin{aligned}
\mathbb{P}^0(\tau_1 \le t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \int_0^{t-1} \frac{|x|}{\sqrt{2\pi y^3}} \exp\left(-\frac{x^2}{2y}\right) \mathrm{d}y\mathrm{d}x \\
&= \frac{1}{\pi} \int_0^{t-1} y^{-3/2} \int_0^{\infty} x \exp\left(\frac{-(y+1)x^2}{2y}\right) \mathrm{d}x\mathrm{d}y \\
&= \frac{1}{\pi} \int_0^{t-1} \frac{1}{\sqrt{y}(y+1)} \mathrm{d}y = \frac{2}{\pi} \arctan\left(\sqrt{t-1}\right).
\end{aligned}$$

(b) Let $\tau_2 = \sup\{s < 1 \mid B_s = 0\}$ and let again $\tau = \inf\{s \ge 0 \mid B_s = 0\}$. We can write

$$\begin{aligned}
\{\tau_2(\omega) \le t\} &= \left\{ \sup\{s < 1 \mid \omega(s) = 0\} \le t \right\} = \left\{ \inf\{s > t \mid \omega(s) = 0\} \ge 1 \right\} \\
&= \left\{ \inf\{s > 0 \mid \omega(s+t) = 0\} \ge 1-t \right\},
\end{aligned}$$

so that

$$\mathbb{1}\{\tau_2 \le t\} = \mathbb{1}\{\tau \ge 1-t\} \circ \theta_t.$$

We now again use the Markov property to calculate the distribution of τ_2:

$$\begin{aligned}
\mathbb{P}^0(\tau_2 \le t) &= \mathbb{E}^0[\mathbb{1}_{\{\tau_2 \le t\}}] = \mathbb{E}^0\left[\mathbb{E}^0[\mathbb{1}_{\{\tau \ge 1-t\}} \circ \theta_t | \mathcal{F}_t]\right] = \mathbb{E}^0\left[\mathbb{E}^{B_t}[\mathbb{1}_{\{\tau \ge 1-t\}}]\right] \\
&= \int_{-\infty}^{\infty} p_t(0,x) \mathbb{P}^x(\tau \ge 1-t) \mathrm{d}x.
\end{aligned}$$

Using the distribution of τ we get

$$\begin{aligned}
\mathbb{P}^0(\tau_2 \le t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{-x^2}{2t}\right) \int_{1-t}^{\infty} \frac{|x|}{\sqrt{2\pi y^3}} \exp\left(-\frac{x^2}{2y}\right) \mathrm{d}y\mathrm{d}x \\
&= \frac{1}{\pi} \int_{1-t}^{\infty} \frac{1}{\sqrt{ty^3}} \int_0^{\infty} x \exp\left(\frac{-(y+t)x^2}{2yt}\right) \mathrm{d}x\mathrm{d}y \\
&= \frac{1}{\pi} \int_{1-t}^{\infty} \frac{yt}{\sqrt{ty^3}(y+t)} \mathrm{d}y = \frac{1}{\pi} \int_{1-t}^{\infty} \sqrt{\frac{(y+t)^2}{yt}} \frac{t}{(y+t)^2} \mathrm{d}y,
\end{aligned}$$

We now use the change of variables $z := t/(y+t)$ to find

$$\mathbb{P}^0(\tau_2 \le t) = \int_0^t \frac{1}{\sqrt{y(1-y)}} \mathrm{d}y = \frac{2}{\pi} \arcsin\left(\sqrt{t}\right).$$